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Gaussian fluctuations of random point measures generated by cooperative sequential adsorption

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ABSTRACT
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Gaussian fluctuations of random point measures generated by cooperative sequential adsorption

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Abstract

A finite number of points is sequentially allocated in a finite domain of a finite-dimensional Euclidean space. The probability distribution of a point depends on all previously allocated points. We consider a situation when this dependence vanishes as the domain is saturated by points. The law of large numbers and the central limit theorem are proved for the generated sequence of random point measures as the number of points goes to infinity.

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1 Introduction and the results

In this paper we study the asymptotic behaviour of random point measures

$$\mu_m = \frac{1}{m}\sum_{i=1}^{m}\delta_{X_i}, \tag{1}$$

generated by random points $X_1, \ldots, X_m$ sequentially allocated in a compact set $D \subset \mathbf{R}^d$. To describe the distribution of the random variables $X_1, \ldots, X_m$ we need some notation. For any point $x \in D$ and a finite non-empty set $\mathbf{y} = (y_1, \ldots, y_n) \in D^n$, $n \geq 1$, we denote by $n(x, \mathbf{y})$ the number of points $y_i \in \mathbf{y}$, such that the distance between x and y_i is not greater than $R(x)$, where $R : D \to \mathbf{R}_+$ is some measurable function. By definition $n(x, \emptyset) = 0$. The number $R(x)$ is called an interaction radius at point x. Let $\{\beta_n(x),\, n \geq 0\}$ be a sequence of measurable positive bounded functions on D. Denote for short $X(k) = (X_1, \ldots, X_k)$, $k \geq 1$, and $X(0) = \emptyset$. Given the set of points $X(k)$ the conditional distribution of point X_{k+1} is specified by the following probability density

$$\psi_{k+1}(x) = \frac{\beta_{n(x,X(k))}(x)}{\alpha(X(k))}, \tag{2}$$

where

$$\alpha(X(k)) = \int\limits_D \beta_{n(u,X(k))}(u)du,$$

the normalizing constant. The joint probability density of $X_1,\dots,X_m$ at points $x_1,\dots,x_m$ is

$$p_m(x_1,\dots,x_m)=\prod_{k=1}^{m}\frac{\beta_{n(x_k,\mathbf{x}_{<k})}(x_k)}{\int_D\beta_{n(x,\mathbf{x}_{<k})}(x)dx}=\prod_{k=1}^{m}\psi(x_k|\mathbf{x}_{<k}),\tag{3}$$

where we denoted for short $\mathbf{x}_{<k}=(x_1,\dots,x_{k-1})$, $k\geq 2$, and $\mathbf{x}_{<1}=\emptyset$ for $k=1$.

Let us give an example where such sets of sequentially allocated random points naturally appear. Consider a spatial birth process $\mathbf{x}(t)$, $t\geq 0$, in D with birth rates defined in terms of functions $\beta_n(x)$, $n\geq 0$, as follows. If the process state at time $t\geq 0$ is $\mathbf{x}$, then the birth rates are $\beta_{n(x,\mathbf{x})}(x)$, $x\in D$, so the total birth rate is $\alpha(\mathbf{x})$ and the time until the next jump is an exponential random variable with mean $\alpha^{-1}(\mathbf{x})$. Assume that $\mathbf{x}(0)=\emptyset$ and consider the set of random variables $X(m)=(X_k,\,k=1,\dots,m)$ formed by the first m points of the process $\mathbf{x}(t)$. It is easy to see that the first point X_1 has the probability distribution specified by the function $\beta_0(x)$ normalized to be a probability density. Conditional on the random variables $X_1,\dots,X_k$, $k\geq 1$, the distribution of X_{k+1} has the probability density given by the formula (2).

If we put $\beta_n(x)=\beta_n$ and $R(x)=R$, then the spatial birth process we have just described is an immediate continuous version of the lattice model of monomer filling with nearest-neighbor cooperative effects ([2]). It is also a particular case of the models of cooperative sequential adsorption widely used in physics and chemistry for modeling of various adsorption processes. For more details and for surveys of the relevant literature we refer to [2], [4], [5] and references therein. Therefore the measures μ_m, $m\geq 1$, (given by the formula (1)) belong to the class of random point measures generated by the spatial processes arising in random sequential packing and deposition problems. For more details on this subject see, for example, [6] and references therein. The typical example is when one sequentially allocates m points in a unit cube. Each point is uniformly distributed in the cube and is accepted with probability depending on configuration of previously accepted points in the ball of radius $1/m$ around the point. Therefore, the interaction radius in those models is inversely proportional to the number of points and it leads to the well-known effect of finite range dependence between points. It is not the case in our model where the interaction radius is a fixed positive function (or constant) regardless of the number of points. This corresponds to the so-called infinite range of interaction or infinite range cooperative effects ([2]).

Apparently, it is not possible to say anything about asymptotic behavior of the sequence (1) unless some additional assumptions on the model parameters are made. Our main assumption is that $\beta_n(x)\to\beta(x)>0$ as $n\to\infty$ uniformly in $x\in D$. The model can be considered as a perturbation of the binomial case. The binomial case is obtained if $\beta_n(x)=\beta(x)$, $x\in D$ for any $n\geq 0$. Intuitively clear that if the β's are bounded from below then the number of points allocated at any fixed domain with positive volume tends to infinity as m goes to infinity. We make it rigorous in Lemma 1.1. Therefore the perturbation vanishes while the domain is saturated by points. The distribution of a "new arrival" becomes "more uniform" and "more independent" on the existing configuration of points provided the domain is sufficiently saturated and the saturation is "sufficiently uniform". In the binomial case (independent allocation) we immediately get Theorems 1.1 and 1.2 (the main results), since in this case they are just the law of large numbers and the central limit theorem for i.i.d. random variables. If β's are not equal to each other then the points are dependent and we arrive at the proof of the law of large numbers and the central limit theorem for the sequence of *dependent* random variables. Though the answers are guessed some care should be taken to assess the weakening of dependence in the tail of the sequence $X(m)$. Note that the random variable X_{k+1} depends on the whole previous history $X_1,\dots,X_k$. If the sequence of functions $\{\beta_n(x),\, n \geq 0\}$ converges, as $n \to \infty$, then the sequence elements with large indexes are becoming asymptotically independent from the past and mutually independent regardless of how close their indexes are. Note that we obtain the central limit theorem (Theorem 1.2) under stronger assumption, namely it is assumed that the sequence of functions $\{\beta_n(x),\, n \geq 0\}$ converges to its limit with exponential rate. This assumption can be considered as an analogy of the assumption of geometric ergodicity that must be made for a Markov chain to obtain the central limit theorem.

Remark. We will denote by the letter C the various constants the particular values of which are immaterial for the proofs. In some cases we will stress dependence of these constants on some parameters that do not depend on the number of points m. By $\mathcal{B}(D)$ we denote the set of real-valued measurable bounded functions on D.

Theorem 1.1 *Assume that* $\inf_{x\in D} R(x) > 0$*, the sequence of positive functions* $\beta_n \in \mathcal{B}(D)$, $n \geq 0$*, is uniformly bounded and converges uniformly as* $n \to \infty$ *to a function* $\beta \in \mathcal{B}(D)$*, such that* $\inf_{x\in D} \beta(x) > 0$*. Then for any function* $f \in \mathcal{B}(D)$

$$J_m(f) = \frac{1}{m}\sum_{i=1}^{m} f(X_i) \to J(f) = \frac{1}{\alpha}\int\limits_D f(x)\beta(x)dx,$$

in probability as $m \to \infty$*, where* $\alpha = \int_D \beta(x)dx$*.*

Theorem 1.2 *Assume that in addition to the assumptions of Theorem 1.1 there exist a positive constant* γ *and a function* $\tau \in \mathcal{B}(D)$ *such that* $\inf_{x\in D} \tau(x) > 0$ *and*

$$|\beta_n(x) - \beta(x)| \leq \tau(x)e^{-\gamma n}, \tag{4}$$

for any $n \geq 0$*. Then the sequence of centered and rescaled random measures* $\sqrt{m}(\mu_m - \mathsf{E}\mu_m)$ *converges as* $m \to \infty$ *to a generalized Gaussian random field on* D *with zero mean and the covariance kernel*

$$G(f,g) = J(fg) - J(f)J(g) = \frac{1}{\alpha}\int\limits_D f(x)g(x)\beta(x)dx - \frac{1}{\alpha^2}\int\limits_D f(x)\beta(x)dx\int\limits_D g(x)\beta(x)dx,$$

for any functions $f, g \in \mathcal{B}(D)$*.*

To prove these theorems we will use Lemmas 1.1-1.5.

Lemma 1.1 *Assume that*

$$0 < \beta_{min} = \inf_n \inf_{x\in D} \beta_n(x) \leq \beta_{max} = \sup_n \sup_{x\in D} \beta_n(x) < \infty,$$

then there exist positive constants λ *and* C *such that for any sufficiently small* $\delta > 0$

$$\mathsf{P}\left\{\inf_{x\in D} n(x, X(m)) \leq m\delta\right\} \leq Ce^{-\lambda m},$$

for all sufficiently large m*.*

Remark. Note that all we need to prove this lemma is positiveness and boundedness of β's.

Corollary 1.1 *Under assumptions of Theorem 1.1 the sequences of random variables*

$$\sup_{x\in D} |\beta_{n(x,X(m))}(x) - \beta(x)| \text{ and } |\alpha(X(m)) - \alpha|$$

tend to 0 *in probability as* $m \to \infty$*, and the sequence* X_m*,* $m \geq 1$*, converges in total variation to the random variable* X *distributed according to the density* $\beta(x)/\alpha$*.*

Let $\mathcal{F}_{k-1}$ be a $\sigma-$algebra generated by the random variables $X_1, \ldots, X_{k-1}$. For any function $f \in \mathcal{B}(D)$ we denote

$$\tilde{J}_k(f) = \mathsf{E}(f(X_k)|\mathcal{F}_{k-1}).$$

Lemma 1.2 *1) Under assumptions of Theorem 1.1 for any function* $f \in \mathcal{B}(D)$ *and for any* $p \geq 1$

$$\mathsf{E}|\tilde{J}_k(f) - J(f)|^p \to 0$$

as $k \to \infty$*.*

2) Under assumptions of Theorem 1.2 there exist positive constants ρ *and* $C = C(f)$ *such that*

$$\mathsf{E}|\tilde{J}_k(f) - J(f)|^p \leq Ce^{-\rho k}$$

as $k \to \infty$*.*

Let Y be a random variable with probability density $\beta(x)/\alpha$. For any function $f \in \mathcal{B}(D)$ and $n \geq 1$ we denote

$$\mathcal{U}_n(f) = \mathsf{E}(f(Y) - \mathsf{E}f(Y))^n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} J(f^i) J^{n-i}(f) \tag{5}$$

Corollary 1.2 *Fix any function* $f \in \mathcal{B}(D)$ *and positive integer* n*. If the assumptions of Theorem 1.2 hold, then*

$$\mathsf{E}\left|\mathsf{E}\left((f(X_k) - \mathsf{E}f(X_k))^n \,|\mathcal{F}_{k-1}\right) - \mathcal{U}_n(f)\right| \leq Ce^{-\rho k}$$

as $k \to \infty$*.*

Lemma 1.3 *Under assumptions of Theorem 1.1 for any functions* $f, g \in \mathcal{B}(D)$ *and indexes* $k \neq j$

$$Cov(f(X_k), g(X_j)) \to 0$$

as $\max(k,j)$ *goes to infinity. If, in addition, the condition (4) holds, then the convergence is exponential in* j *if* k *is fixed and the convergence is exponential in* $\min(k,j)$ *if both* k *and* j *tend to infinity.*

Denote $\xi_k(f) = f(X_k) - \mathsf{E}f(X_k)$.

Lemma 1.4 *Fix a set of functions* $g_1, \ldots, g_k \in \mathcal{B}(D)$ *and a set of positive integers* $r_1, \ldots, r_k$ *and let* $n = r_1 + \cdots r_k$*. If the condition (4) holds, then there exist constants* $C = C(k,g)$ *depending on* k *and functions* g'*s such that*

$$\left|\mathsf{E}\prod_{v=1}^{k} \xi_{i_v}^{r_k}(g_v) - \prod_{v=1}^{k} \mathcal{U}_{r_k}(g_v)\right| \leq C^n \sum_{v=1}^{k} e^{-\rho i_v},$$

for all sufficiently large indexes $i_1 < \ldots < i_k$*,* $k \geq 1$*, where the constant* ρ *is determined in Lemma 1.2 and* $\mathcal{U}_{r_k}(g_v)$ *is defined by the formula (5).*

Lemma 1.5 *1) Under assumptions of Theorem 1.1 for any function $f \in \mathcal{B}(D)$*

$$\frac{1}{m}\sum_{k=1}^{m} \mathsf{E} f(X_k) \to J(f),$$

as $m \to \infty$.

2) If, in addition, the condition (4) holds, then for any functions $f, g \in \mathcal{B}(D)$

$$mCov(J_m(f), J_m(g)) \to \frac{1}{\alpha}\int\limits_D f(x)g(x)\beta(x)dx - \frac{1}{\alpha^2}\int\limits_D f(x)\beta(x)dx \int\limits_D g(x)\beta(x)dx$$

as $m \to \infty$.

2 Proof of Theorem 1.1

For any function $f \in \mathcal{B}(D)$ we have by Lemma 1.5 that

$$\frac{1}{m}\sum_{k=1}^{m} \mathsf{E} f(X_k) \to J(f).$$

By Chebyshev inequality we have for any $\varepsilon > 0$ that

$$\mathsf{P}\left\{\left|\sum_{k=1}^{m} f(X_k) - \mathsf{E} f(X_k)\right| \geq \varepsilon m\right\} \leq \frac{1}{\varepsilon^2 m^2}\sum_{k,j=1}^{m} Cov(f(X_k), f(X_j)).$$

By Lemma 1.3 $Cov(f(X_k), f(X_j))$ converges to 0 for any $k \neq j$ as $max(k, j) \to \infty$, therefore the right-hand side of the preceding display vanishes as $m \to \infty$. Theorem is proved.

3 Proof of Theorem 1.2

It is well known (see, for instance, [1]) that to prove the theorem it suffices to prove that for any function $f \in \mathcal{B}(D)$ the sequence of random variables

$$S_m(f) = \frac{1}{\sqrt{m}}\sum_{k=1}^{m}(f(X_k) - \mathsf{E} f(X_k))$$

converges weakly to a Gaussian random variable with mean zero and the variance $J(f^2) - J^2(f)$ as $m \to \infty$. Fix some $0 < \delta < 1/2$ and write

$$S_m(f) = \frac{1}{\sqrt{m}}\sum_{k=1}^{m^\delta - 1}(f(X_k) - \mathsf{E} f(X_k)) + \sqrt{\frac{m - m^\delta}{m}}\frac{1}{\sqrt{m - m^\delta}}\sum_{k=m^\delta}^{m}(f(X_k) - \mathsf{E} f(X_k)).$$

The first sum above can be bounded by $2\|f\|_\infty m^{\delta - 1/2}$ and vanishes as $m \to \infty$. Therefore it suffices to prove that the sequence of random variables

$$\tilde{S}_m(f) = \frac{1}{\sqrt{m - m^\delta}}\sum_{k=m^\delta}^{m}(f(X_k) - \mathsf{E} f(X_k))$$

converges to a Gaussian random variable with mean zero and the variance $J(f^2)-J^2(f)$ as $m\to\infty$. We are going to show that $\mathcal{K}_{mn}(f)$ the nth cumulant (semiinvariant) of $\tilde{S}_m(f)$ converges as $m\to\infty$ to the cumulant of a Gaussian random variable with zero mean and the variance $J(f^2)-J^2(f)$. It is well known that the convergence of cumulants implies the weak convergence (see, for instance, Lemma 3 in [8]). We have actually proved in Lemma 1.5, part 2), that $\mathcal{K}_{m2}(f)\to J(f^2)-J^2(f)$ as $m\to\infty$. We need to prove that $\mathcal{K}_{mn}(f)\to 0$ as $m\to\infty$ for $n\geq 3$. Recall that the cumulants $\mathcal{K}_{mn}(f)$, $n\geq 1$, are defined as the Taylor coefficients of the logarithm of the characteristic function

$$\log \mathsf{E} e^{it\tilde{S}_m(f)} = \sum_{n=1}^{\infty} \mathcal{K}_{mn}(f)\frac{(it)^n}{n!},\quad t\in\mathbf{R}. \tag{6}$$

Each cumulant $\mathcal{K}_{mn}(f)$, $n\geq 1$, is well defined since the random variable $\tilde{S}_m(f)$ has all finite moments and can be presented as a finite linear combination of the products of moments ([3])

$$\mathcal{K}_{mn}(f) = \sum_{k=1}^{n}(-1)^{k-1}(k-1)!\sum_{r_1,\dots,r_k}\prod_{j=1}^{k}\mathsf{E}\tilde{S}_m^{r_j}(f), \tag{7}$$

where the second sum is over all sets of positive integers $\{r_1,\dots,r_k\}$ such that $r_1+\cdots+r_k=n$.

Let $Y_i, i\geq 1$, be the collection of independent random variables in D having the common probability density $\beta(x)/\alpha$. Denote

$$S_{0,m}(f) = \frac{1}{\sqrt{m_\delta}}\sum_{k=1}^{m_\delta}(f(Y_k)-\mathsf{E}f(Y_k)),$$

where we denoted $m_\delta = m - m^\delta$. We are going to show that for a fixed set of positive indexes $r_1,\dots,r_k$, such that $r_1+\cdots+r_k=n$ the following expansion holds

$$\prod_{j=1}^{k}\mathsf{E}\tilde{S}_m^{r_j}(f) = \prod_{j=1}^{k}\mathsf{E}S_{0,m}^{r_j}(f) + \zeta_m(r_1,\dots,r_k,f), \tag{8}$$

where

$$|\zeta_m(r_1,\dots,r_k,f)| \leq C(n)m^{\delta+n/2}e^{-\rho m^\delta}.$$

For the simplicity of notation we prove the expansion (8) for the particular case $k=1, r_1=n$, for other cases the proof is similar. It is easy to see that

$$\mathsf{E}\tilde{S}_m^n(f) = m_\delta^{-n/2}\sum_{t_1,\dots,t_p}\ \sum_{m_\delta\leq i_1<\dots<i_p\leq m}\mathsf{E}\prod_{v=1}^{p}\xi_{i_v}^{t_v}(f),$$

where the first sum is over all sets of positive integers t_i, $i=1,\dots,p$, such that $t_1+\cdots+t_p=n$. We get the expansion (8) if we put

$$\zeta_m(n,f) = m_\delta^{-n/2}\sum_{t_1,\dots,t_p}\ \sum_{m_\delta\leq i_1<\dots<i_p\leq m}\left(\mathsf{E}\prod_{v=1}^{p}\xi_{i_v}^{t_v}(f) - \prod_{v=1}^{p}\mathcal{U}_{t_v}(f)\right).$$

Indeed, by Lemma 1.4 we have that

$$|\zeta_m(n,f)| \le m_\delta^{-n/2} \sum_{t_1,\dots,t_p} \sum_{m_\delta \le i_1 < \dots < i_p \le m} C^n \sum_{v=1}^{p} e^{-\rho i_v}. \tag{9}$$

It is easy to see that for any fixed set of positive integers $t_1, \dots, t_p$ in the first sum we can bound

$$\begin{aligned} m_\delta^{-n/2} \sum_{m_\delta \le i_1 < \dots < i_p \le m} C^n \sum_{v=1}^{p} e^{-\rho i_v} &\le C(n,\rho,\delta) m^{(\delta-1/2)n} (m - m^\delta)^{p-1} e^{-\rho m^\delta} \\ &\le C(n,\rho,\delta) m^{\delta+n/2} e^{-\rho m^\delta}. \end{aligned}$$

Note that the first sum in (9) contains the number of terms depending only on n. Therefore we have that

$$|\zeta_m(n,f)| \le C(n) m^{\delta+n/2} e^{-\rho m^\delta}.$$

For other sets of integers $r_1, \dots, r_k$ the expansion (8) can be obtained in the same way.

Going back to (7) we get that

$$\mathcal{K}_{mn}(f) = \mathcal{K}_{mn}^{(0)}(f) + \sum_{k=1}^{n} (-1)^{k-1} (k-1)! \sum_{r_1,\dots,r_k} \zeta_m(r_1, \dots, r_k, f), \tag{10}$$

where $\mathcal{K}_{mn}^{(0)}(f)$ is nth cumulant of the random variable $S_{0,m}(f)$. Because of the independence we have that

$$\mathcal{K}_{mn}^{(0)}(f) \sim m_\delta^{-n/2+1} \to 0$$

for any $n > 2$ as $m \to \infty$. It remains to note that

$$\left| \sum_{k=1}^{n} (-1)^{k-1} (k-1)! \sum_{r_1,\dots,r_k} \zeta_m(r_1, \dots, r_k, f) \right| \le C(n) n! m^{\delta+n/2} e^{-\rho m^\delta} \to 0,$$

as $m \to \infty$. The theorem is proved.

4 Proofs of Lemmas 1.1-1.5

Without loss of generality and for the simplicity of notation we assume throughout the proofs of Lemmas 1.1-1.2, that the functions $\beta_n(x)$, $n \ge 0$, $\beta(x)$ and $R(x)$ are constant, i.e. $\beta_n(x) = \beta_n$, $n \ge 0$, $\beta(x) = \beta$ and $R(x) = R$, $x \in D$. Also, we assume in these proofs that D is a $d-$dimensional unit cube. Therefore the limit distribution for the sequence of random variables X_m, $m \ge 1$, is just a uniform distribution in the unit cube. Modifications for the general case are obvious.

Proof of Lemma 1.1. Fix sufficiently large $l \in \mathbf{Z}_+$ such that $p(l) = l^{-d}(\beta_{min}/\beta_{max}) < 1$ and, also $1/l < R/4$. Let $\{Q_i,\, i = 1, \dots, l^d\}$ be the set of non-overlapping cubes of size $1/l$, such that

$$D = \bigcup_i Q_i.$$

Denote by ξ_{mi} a number of points $X_1, \ldots, X_m$ falling in the cube Q_i. Take a point $x \in D$ and let $x \in Q_i$ for some i. It is easy to see that

$$n(x, X(m)) \geq \xi_{mi} \geq \min_j \xi_{mj}, \tag{11}$$

since $Q_i \subset B(x, R)$. Fix some $0 < \delta < p(l)$ and denote $A_m = \{\min_i \xi_{mi} > m\delta\}$. The equation (11) implies that in order to prove the lemma it suffices to bound exponentially the probability $\mathsf{P}\{\overline{A}_m\} \to 0$ as $m \to \infty$, where

$$\overline{A}_m = \bigcup_i \{\xi_{mi} \leq m\delta\}.$$

It is obvious that

$$\mathsf{P}\{\overline{A}_m\} \leq l^d \max_i \mathsf{P}\{\xi_{mi} \leq m\delta\}.$$

The formula (2) yields that

$$\mathsf{P}\{X_k \in Q_i | X(k-1)\} = \frac{\int_{Q_i} \beta_{n(u,X(k-1))} du}{\int_D \beta_{n(u,X(k-1))} du}.$$

This conditional probability can be bounded from below by $p(l)$ uniformly in sequences $X(k-1)$. Therefore the unconditional probability $\mathsf{P}\{X_k \in Q_i\}$ is also bounded from below by the same constant for any $k \geq 1$. Using the well-known coupling construction (see, for instance, Theorem I.5.1 in [7]) we can construct on the same probability space the random variable ξ_{mi} and the binomial random variable $\tilde{\xi}_{mi}$ with m trials and with $p(l)$ the probability of success such that ξ_{mi} stochastically dominates $\tilde{\xi}_{mi}$. So, we have that $\mathsf{P}\{\xi_{mi} \leq m\delta\} \leq \mathsf{P}\{\tilde{\xi}_{mi} \leq m\delta\}$. The standard bounds of the probabilities of large deviations for the sums of i.i.d. random variables give us that for any $0 < \delta < p(l)$

$$\mathsf{P}\{\tilde{\xi}_{mi} \leq m\delta\} \leq Ce^{-\lambda m},$$

with some positive constants C and λ. Therefore

$$\mathsf{P}\left\{\inf_{x \in D} n(x, X(m)) \leq m\delta\right\} \leq l^d \max_i \mathsf{P}\{\xi_{mi} \leq m\delta\} \leq Cl^d e^{-\lambda m}.$$

Lemma is proved.

Proof of Corollary 1.1. The part 1) of Corollary 1.1 is an immediate implication of Lemma 1.1 and the convergence of the β's. Indeed, for any $\varepsilon > 0$ we have $\sup_{x \in D} |\beta_{n(x,X(m))}(x) - \beta(x)| < \varepsilon$ as soon as $\inf_{x \in D} n(x, X(m)) > n(\varepsilon)$, for some $n(\varepsilon)$.

By the equation (2) the unconditional density of the random variable X_{k+1} at a point u is

$$\mathsf{E}\psi(u|X(k)) = \mathsf{E}\frac{\beta_{n(u,X(k))}(u)}{\alpha(X(k))}.$$

The integrand in this mean is bounded and converges in probability to $\beta(u)/\alpha$ as $k \to \infty$ by the part 1) of Corollary. Therefore, $\mathsf{E}\psi(u|X(k)) \to \beta(u)/\alpha$ for any $u \in D$ as $k \to \infty$. It is well known that the point-wise convergence of densities implies the convergence in total variation. Corollary is proved.

Proof of Lemma 1.2. We start with the part 1). Fix arbitrary $\varepsilon > 0$ and $\delta > 0$ and denote

$$B_{k,\varepsilon} = \left\{ \sup_{x \in D} |\beta_{n(x,X(k))} - \beta| < \varepsilon \right\}.$$

It is easy to see that

$$\tilde{J}_k(f) = \frac{1}{\alpha(X(k-1))} \int\limits_D f(x)\beta_{n(x,X(k-1))} dx, \ k \geq 1.$$

One can write

$$\begin{aligned} \mathsf{E}|\tilde{J}_k(f) - J(f)|^p &= \mathsf{E}\left| \frac{\int_D f(x)\beta_{n(x,X(k-1))}dx}{\alpha(X(k-1))} - J(f) \right|^p I_{\{B_{k,\varepsilon}\}} \\ &+ \mathsf{E}\left| \frac{\int_D f(x)\beta_{n(x,X(k-1))}dx}{\alpha(X(k-1))} - J(f) \right|^p I_{\{\overline{B}_{k,\varepsilon}\}} \\ &= S_1 + S_2 \end{aligned}$$

Note that

$$S_1 = \mathsf{E}\left| J(f)\left(\frac{\beta}{\int_D \beta_{n(x,X(k-1))}dx} - 1 \right) + \frac{\int_D f(x)(\beta_{n(x,X(k-1))} - \beta)dx}{\int_D \beta_{n(x,X(k-1))}dx} \right|^p I_{\{B_{k,\varepsilon}\}}.$$

By Minkovski inequality

$$\begin{aligned} S_1^{1/p} &\leq |J(f)| \left(\mathsf{E}\left| \frac{\beta}{\int_D \beta_{n(x,X(k-1))}dx} - 1 \right|^p I_{\{B_{k,\varepsilon}\}} \right)^{1/p} \\ &+ \left(\mathsf{E}\left| \frac{\int_D f(x)(\beta_{n(x,X(k-1))} - \beta)dx}{\int_D \beta_{n(x,X(k-1))}dx} \right|^p I_{\{B_{k,\varepsilon}\}} \right)^{1/p} \end{aligned}$$

Therefore

$$S_1 < 2^p \|f\|_\infty^p \mathsf{E}\left(\frac{\int_D |\beta_{n(x,X(k-1))} - \beta|dx}{\int_D \beta_{n(x,X(k-1))}dx} \right)^p I_{\{B_{k,\varepsilon}\}} < C\varepsilon^p. \tag{12}$$

By Lemma 1.1 there exists $k(\delta,\varepsilon)$ such that $\mathsf{P}\{B_{k,\varepsilon}\} > 1 - \delta$, if $k > k(\delta,\varepsilon)$. Therefore if $k > k(\delta,\varepsilon)$, then one can bound

$$S_2 \leq \|f\|_\infty^p \left(\frac{\beta_{max}}{\beta_{min}} + 1 \right)^p \mathsf{P}\{\overline{B}_{k,\varepsilon}\} < C\delta. \tag{13}$$

Combining bounds (12) and (13) we get that if $k > m(\delta,\varepsilon) + 1$, then

$$\|\tilde{J}_k(f) - J(f)\|_{L^p}^p < C(\varepsilon^p + \delta)$$

for some constant $C > 0$. Therefore L^p-convergence of $\tilde{J}_k(f)$ to $J(f)$ is proved for any $p > 1$, since ε and δ were taken arbitrary. The part 1) of the lemma is proved.

Let us prove the part 2) of Lemma 1.2. Fix sufficiently small positive δ for which Lemma 1.1 holds. Denote

$$A_{k,\delta} = \left\{ \inf_{x \in D} n(x, X(k)) \geq k\delta \right\}.$$

Let us write again

$$\mathsf{E}|\tilde{J}_k(f) - J(f)|^p = S_1^{'} + S_2^{'},$$

where now

$$S_1^{'} = \mathsf{E}\left|\frac{\int_D f(x)\beta_{n(x,X(k-1))}dx}{\alpha(X(k-1))} - J(f)\right|^p I_{\{A_{k,\delta}\}}$$

and

$$S_2^{'} = \mathsf{E}\left|\frac{\int_D f(x)\beta_{n(x,X(k-1))}dx}{\alpha(X(k-1))} - J(f)\right|^p I_{\{\overline{A}_{k,\delta}\}}.$$

Similar to the bounds (12) and (13) we have

$$S_1^{'} < 2^p \|f\|_\infty^p \mathsf{E}\left(\frac{\int_D |\beta_{n(x,X(k-1))} - \beta| dx}{\int_D \beta_{n(x,X(k-1))}dx}\right)^p I_{\{A_{k,\delta}\}}, \tag{14}$$

and

$$S_2^{'} \le \|f\|_\infty^p \left(\frac{\beta_{max}}{\beta_{min}} + 1\right)^p \mathsf{P}\{\overline{A}_{k,\delta}\}. \tag{15}$$

Using exponential convergence of $\beta_n^{\prime}$s we obtain

$$S_1^{'} \le \left(\frac{2\|f\|_\infty}{\beta_{min}}\right)^p e^{-\gamma\delta pk}, \tag{16}$$

and by Lemma 1.1 we get

$$S_2^{'} \le \|f\|_\infty^p \left(\frac{\beta_{max}}{\beta_{min}} + 1\right)^p e^{-\lambda k} \tag{17}$$

for some $\lambda > 0$. The equations (16) and (17) yield that

$$\mathsf{E}|\tilde{J}_k(f) - J(f)|^p \le Ce^{-k\min(\lambda,\gamma\delta p)},$$

and the second part of the lemma is proved.

Proof of Corollary 1.2. The statement of Corollary directly follows from Lemma because we have by the binomial formula that

$$\mathsf{E}\left((f(X_k) - \mathsf{E}f(X_k))^n \,|\mathcal{F}_{k-1}\right) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\tilde{J}_k(f^i)J_k^{n-i}(f).$$

Proof of Lemma 1.3. Assume for definiteness that $k \le j$. We can write

$$\mathsf{E}f(X_k)g(X_j) = \mathsf{E}\left(f(X_k)\mathsf{E}\left(g(X_j)|\mathcal{F}_{j-1}\right)\right). \tag{18}$$

1) Assume first that k is fixed and j goes to infinity. By Lemma 1.2 $\tilde{J}_j(g) = \mathsf{E}\left(g(X_j)|\mathcal{F}_{j-1}\right)$ converges in L^2 to $J(g)$ as $j \to \infty$. It implies that $\mathsf{E}f(X_k)g(X_j) \to J(g)\mathsf{E}f(X_k)$ as $j \to \infty$ because of the continuity of the inner product in the Hilbert space. Lemma 1.2 yields that $\mathsf{E}f(X_k) \to J(f)$, as $k \to \infty$. Therefore $\mathsf{E}f(X_k)\mathsf{E}g(X_j) \to J(g)\mathsf{E}f(X_k)$ as $j \to \infty$, so we get the result in this case.

2) Assume now that both k and j go to infinity. Again, by Lemma 1.3 $\mathsf{E}f(X_k)\mathsf{E}g(X_j) \to J(g)J(f)$ as $j \to \infty$. By the equation (18)

$$\mathsf{E}f(X_k)g(X_j) = \mathsf{E}f(X_k)(\tilde{J}_j(g) - J(g)) + J(g)\mathsf{E}f(X_k).$$

The first term in the right side of the preceding equation is bounded by $\|f\|_\infty \|\tilde{J}_j(g) - J(g)\|_{L^1}$ and it goes to zero as $j \to \infty$ by Lemma 1.2. The second term $J(g)\mathsf{E}f(X_k)$ also tends to $J(g)J(f)$ by Lemma 1.2.

It is easy to see that in both cases the convergence is exponential in j in the first case and in k in the second one if the condition (4) holds.

Proof of Lemma 1.4. Conditioning on $\mathcal{F}_{i_k-1}$ we have

$$\mathsf{E}\prod_{v=1}^{k} \xi_{i_v}^{r_v}(g_v) = \mathsf{E}\left(\prod_{v=1}^{k-1} \xi_{i_v}^{r_v}(g_v)\mathsf{E}\left((g_{i_k}(X_{i_k}) - \mathsf{E}g_{i_k}(X_{i_k}))^{r_k}\,|\mathcal{F}_{i_k-1}\right)\right),$$

so

$$\begin{aligned}\mathsf{E}\prod_{v=1}^{k} \xi_{i_v}^{r_v}(g_v) &= \mathsf{E}\prod_{v=1}^{k-1} \xi_{i_v}^{r_v}(g_v)\mathcal{U}_{r_k}(g_{i_k}) \\ &\quad + \mathsf{E}\prod_{v=1}^{k-1} \xi_{i_v}^{r_v}(g_v)\left(\mathsf{E}\left((g_{i_k}(X_{i_k}) - \mathsf{E}g_{i_k}(X_{i_k}))^{r_k}\,|\mathcal{F}_{i_k-1}\right) - \mathcal{U}_{r_k}(g_{i_k})\right)\end{aligned}$$

Using Corollary 1.2 and boundedness of g's we can bound the second term in the right-hand side of the preceding display as follows

$$\left|\mathsf{E}\prod_{v=1}^{k-1} \xi_{i_v}^{r_v}(g_v)\left(\mathsf{E}\left((g_{i_k}(X_{i_k}) - \mathsf{E}g_{i_k}(X_{i_k}))^{r_k}\,|\mathcal{F}_{i_k-1}\right) - \mathcal{U}_{r_k}(g_{i_k})\right)\right| \le C^n(g)e^{-\rho i_k}.$$

Repeating this procedure sequentially for the indexes $i_{k-1}, \ldots, i_1$ in the product $\mathsf{E}\prod_{v=1}^{k-1} \xi_{i_v}^{r_v}(g_v)\mathcal{U}_{r_k}(g_{i_k})$ we finish the proof.

Proof of Lemma 1.5. By Lemma 1.2 we have that $\mathsf{E}f(X_k) \to J(f)$, as $k \to \infty$. Fix an arbitrary $\varepsilon > 0$ and let $k(\varepsilon)$ be such that $|\mathsf{E}f(X_k) - J(f)| \le \varepsilon$ as $k > k(\varepsilon)$. It is easy to see that

$$\left|\frac{1}{m}\sum_{k=1}^{m} \mathsf{E}f(X_k) - J(f)\right| \le 2\frac{k(\varepsilon)}{m}\|f\|_\infty + \frac{m - k(\varepsilon)}{m}\varepsilon.$$

The first term in the right side of the preceding equation goes to 0 as $m \to \infty$, the second is less than ε. Therefore we get the result since ε is arbitrary.

Let us prove the part 2) of the lemma. Let us fix some positive $\delta < 1/2$ and write

$$mCov(J_m(f), J_m(g)) = \frac{1}{m} \sum_{k=[m^\delta]+1}^{m} Cov(f(X_k), g(X_k))$$
$$+ \frac{1}{m} \sum_{k,j=1}^{[m^\delta]} Cov(f(X_k), g(X_j))$$
$$+ \frac{1}{m} \sum_{k \neq j > [m^\delta]} Cov(f(X_k), g(X_j)).$$

By Lemma 1.3 we have that $Cov(f(X_k), g(X_k)) \to J(fg) - J(f)J(g)$, as $k \to \infty$, therefore the first sum in the preceding display converges to $J(fg) - J(f)J(g)$ by the argument we have just used for the means. The second sum can be bounded by $Cm^{2\delta-1}$ and vanishes as $m \to \infty$. The third sum can be bounded by $\max_{k,j>[m^\delta]} Cov(f(X_k), g(X_j))m$. By the assumption (4) and Lemma 1.3 $\max_{k,j>[m^\delta]} Cov(f(X_k), g(X_j))$ decays exponentially as $m \to \infty$ and the proof is over.

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